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An arbitrary Lagrangian Eulerian (ALE) formulation for free surface flows using the characteristic-based split (CBS) scheme

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SUMMARY

An arbitrary Lagrangian Eulerian (ALE) method for non-breaking free surface flow problems is presented. The characteristic-based split (CBS) scheme has been employed to solve the ALE equations. A simple mesh smoothing procedure based on coordinate averaging (Laplacian smoothing) is employed in the calculations. The mesh velocity is calculated at each time step and incorporated as part of the scheme. Results presented show an excellent agreement with the available experimental data. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: free surface; ALE method; incompressible; CBS scheme

1. INTRODUCTION

In many problems of practical importance a free surface will occur in the fluid (always liquid). In general the position of such a free surface is not known and the main problem is that of determining it. Typical problems of free surfaces include flow over and under water control structures, flow around ships, to industrial processes such as filling of moulds. All these situations deal with a fluid which is incompressible.

There are several ways of dealing with free surface flows. We broadly classify them into three categories. They are (i) pure Lagrangian methods (ii) Eulerian methods and (iii) arbitrary Lagrangian Eulerian (ALE) methods.

In Lagrangian methods we need to employ the equations for the fluid particles whose position is changing continuously in time [1–4]. Such Lagrangian methods almost always are

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used in the study of solid mechanics but are relatively seldom applied in fluid dynamics. This is due to either that in most unsteady fluid dynamics problems very large deformation occurs or unsteady state is not important. There is an immediate advantage of Lagrangian formulation in the fact that convective acceleration is non-existent and the problem is immediately self-adjoint. Further, for problems in which free surface occurs it allows the free surface to be continuously updated and maintained during the fluid motion.

In Eulerian methods for which the boundaries of the fluid motion are fixed in position and so indeed are any computational meshes. For free surface problems an immediate difficulty arises as the position of the free surface is not known *a priori*. The numerical method will therefore have to include an additional algorithm to trace the free surface positions [5–15]. This is normally carried out by solving a convection equation for free surface height on the free boundary.

With both Lagrangian and Eulerian methods certain difficulties and advantages occur and on occasion it is possible to provide a better alternative, which attempts to secure the best features of both Lagrangian and Eulerian description by combining these. Such methods are known as ALE methods [16–29].

The topic of interest in this paper is the ALE method for non-breaking free surface flow problems. The ALE description of a fluid is discussed in detail by many (see Reference [28] for a detailed mathematical description). However, implementation of ALE method for free surface flow calculations follows several different approaches [16–29]. The major difference between existing finite-element-based papers is the calculation of mesh velocity. Many of these papers mention very little about the mesh velocity and its integration with the main algorithm. Thus, in this paper we have attempted to provide a step-by-step procedure which is easy to implement.

The ALE method developed in this paper is based on estimating a grid velocity at each time interval on nodes from a mesh smoothing algorithm. The mesh smoothing algorithm employed is based on determining the position of a node by simple averaging of the coordinates of the surrounding nodes. Such a smoothing is applied during each time interval. It should be noted, however, that any other mesh smoothing or mesh regeneration procedure can be integrated as part of the proposed algorithm.

In Section 2 we provide equations of ALE description of a fluid followed by in Section 3 a brief discussion of the characteristic-based split (CBS) scheme. Section 4 gives a brief implementation procedure of the ALE approach for free surface flow problems and Section 5 gives two simple examples, to demonstrate the use of the proposed ALE approach and the CBS scheme. Finally, Section 6 draws some conclusions.

2. ARBITRARY LAGRANGIAN EULERIAN (ALE) FORMULATION

The ALE settings for incompressible flows is written as

continuity

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1}$$

and *momentum*

$$\frac{\partial u_i}{\partial t} + (u_i - u_{gi})\frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} - g_i \tag{2}$$

where ρ is the density, t is the time, u_i are the velocity components, p is the pressure and g_i is the acceleration due to gravity. The subscript g in the above equation indicates the grid velocity. If the explicit form of solution is preferred then an artificial compressibility term is added to Equation (1) and the continuity equation becomes

$$\frac{1}{\beta^2}\frac{\partial p}{\partial t} + \rho\frac{\partial u_i}{\partial x_i} = 0 \tag{3}$$

here β is an artificial compressible wave speed.

It is easy to observe that Equation (2) becomes a Lagrangian equation if grid velocity u_{gi} is equal to fluid velocity u_i and it becomes an Eulerian equation if $u_{gi} = 0$. Thus, this formulation may be considered to be one which encompasses all the three methodologies of free surface flow mentioned previously. The problem definition will be complete with the specification of appropriate boundary conditions. In this study, pressure values are assumed to be zero on the free surface and all walls are assumed to be slip walls.

3. ALE IMPLEMENTATION

The implementation of ALE method varies depending on the problem of interest. The major point which deserves attention is allotting appropriate nodal mesh velocities. In simple problems it may be possible to impose a grid velocity on the nodes *a priori*. However, generalizing such an approach is not possible. It is therefore necessary to have a mesh smoothing/regeneration algorithm as part of the solution procedure.

There are several mesh smoothing procedures available for triangular and tetrahedral elements. One such procedure was introduced by Giuliani [30] in which a function constructed from measures of distortion and squeeze is minimized. This procedure works well for domains with fixed boundaries. Several improvements have been later carried out by many authors [31]. A variable smoothing method based on a combination of Laplacian and Winslow's method was introduced by Hermansson and Hansbo [32], which preserves the element stretching. The objectives of many of these smoothing procedures is to keep the number of elements the same through out the calculation. It is therefore obvious that these algorithms are limited to low amplitude free surface waves. For larger amplitudes it may be necessary to use remeshing of the whole or part of the domain.

We have also used a simple smoothing procedure widely employed in mesh generation algorithms to improve element quality. Here, position of a node inside a domain is recalculated as an average of the coordinates of the surrounding nodes. Depending on the requirement this smoothing procedure can be employed several times within a single time step.

It is standard practice to split an ALE algorithm into three phases. They are (1) Lagrangian solution (2) Mesh rezoning action and (3) Eulerian calculation [2]. However, in practice such a distinction may not be necessary. We do not divide the ALE method into three stages. The proposed algorithm has the following steps within each time interval:

```
(1) apply mesh smoothing
(2) calculate mesh velocities
(3) estimate the nodal variables by solving the governing equations
    (velocities and pressure)
(4) move the nodes to new positions using speeds calculated at step 3
(5) go to next time step
```

The relations used for the mesh velocities at step 2 and nodal positions at step 4 are given in the following section.

4. SOLUTION PROCEDURE

The three steps of the CBS procedure can be summarized as [33–38]

Step 1: Intermediate momentum

$$\Delta\tilde{u}_i = \tilde{u}_i - u_i^n = \Delta t\left[-(u_i - u_{gi})\frac{\partial u_i}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} - g_i\right]^n \tag{4}$$

where $u_i^n = u_i(t_n)$; $\Delta t = t^{n+1} - t^n$ and ˜ indicates an intermediate quantity.

Step 2: Pressure

$$\left(\frac{1}{\beta^2}\right)^n \Delta p = \left(\frac{1}{\beta^2}\right)^n (p^{n+1} - p^n) = -\Delta t\left[\rho\frac{\partial u_i^n}{\partial x_i} + \theta_1\rho\frac{\partial \Delta u_i^*}{\partial x_i} - \Delta t\theta_1\left(\frac{\partial^2 p^n}{\partial x_i \partial x_i} + \theta_2\frac{\partial^2 \Delta p}{\partial x_i \partial x_i}\right)\right] \tag{5}$$

Selection of artificial compressible wave speed β for an explicit scheme is discussed in Reference [33].

Step 3: Momentum correction

$$\Delta u_i = u_i^{n+1} - u_i^n = \Delta u_i^* - \frac{1}{\rho}\Delta t\,\frac{\partial p}{\partial x_i}^{n+\theta_2} \tag{6}$$

The standard Galerkin approximation can now be applied to all the three steps. For full details refer to References [33–38].

For a fully explicit form $0.5 \geqslant \theta_1 \geqslant 1.0$ and $\theta_2 = 0$ and β value is finite. However, for a semi-implicit form $0.5 \geqslant \theta_1 \geqslant 1.0$ and $0.5 \geqslant \theta_2 \geqslant 1.0$ and β is infinity and thus left-hand side of Equation (5) is equal to zero. In a semi-implicit form, therefore, step 2 is an implicit calculation step for pressure.

If one prefers to use the explicit scheme, a dual time stepping procedure is helpful to carry out calculations in real time. In such a procedure a real time step is added to step 3 of the scheme [33].

The Lagrangian movement of the coordinates are facilitated using the following relation after every real time step [2]:

$$x_i^{n+1} = x_i^n + \tfrac{1}{2}\,\Delta t(u_i^{n+1} + u_i^n) \tag{7}$$

The grid velocity on a node can obviously be calculated from the above equation

$$u_{gi}^{n+1} = \frac{2}{\Delta t}\,(x_i^{n+1} - x_i^n) - u_{gi}^n \tag{8}$$

Here Δt is the real time step. The steps in an ALE procedure using the CBS scheme is summarized as

```
Do i = 1, number of time steps
  Step1: Mesh smoothing using nodal averaging (Section 3)
  Step2: Mesh velocity calculation (Equation (8))
  Step3: Intermediate velocity (Equation (4))
  Step4: Pressure calculation (Equation (5))
  Step5: Velocity correction (Equation (6))
  Step6: Nodal displacement (Equation (7))
enddo !i
```

If a pure Lagrangian calculation is preferred the convection and stabilizing terms in Equation (4) are switched off to save computing time.

5. EXAMPLES

5.1. Model broken dam

The first example problem considered is a standard benchmark problem of model broken dam as shown in Figure 1. This problem is solved as a pure Lagrangian problem with $u_i = u_{gi}$. Here, no mesh smoothing is employed.

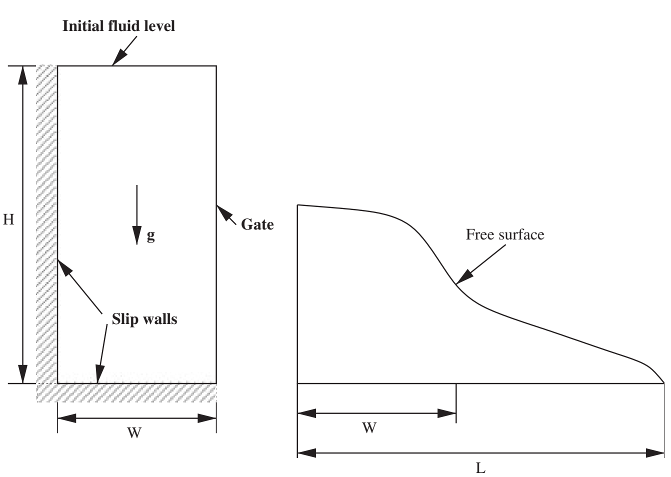

Figure 1. Broken dam problem. Problem definition and schematic of the free surface.

As seen in Figure 1 the problem consists of two slip walls on which slip boundary conditions are applied (normal velocity zero). The initial fluid position is as shown in Figure 1 (left) with velocities at all nodes equal to zero. The dimensions of the dam are: $H = 7$ and $W = 3.5$. The gravity was assumed to act with a magnitude equal to unity. The viscosity was assumed to be 10^{-2}. The initial conditions are given as standard static conditions with zero values for velocity components and hydrostatic value for pressure.

At $t = 0$, the gate was opened and the fluid from the tank was allowed to flow freely. The quantity of interest is the extreme horizontal free surface position L as shown in Figure 1 (right). The unstructured mesh used consists of 339 nodes and 604 elements. The semi-implicit form of the CBS scheme was employed to solve this problem.

Figures 2 and 3 give the meshes and contours of variables at time levels 2.0 and 5.0. As seen the results are generally smooth all over the domain although the mesh is very stretched at $t = 5.0$. The pressure contours are free of oscillations, which shows the excellent pressure stabilization properties of the CBS scheme.

Figure 4 shows the comparison of extreme horizontal position of the free surface with the experimental data. As seen the numerical results are in excellent agreement with the

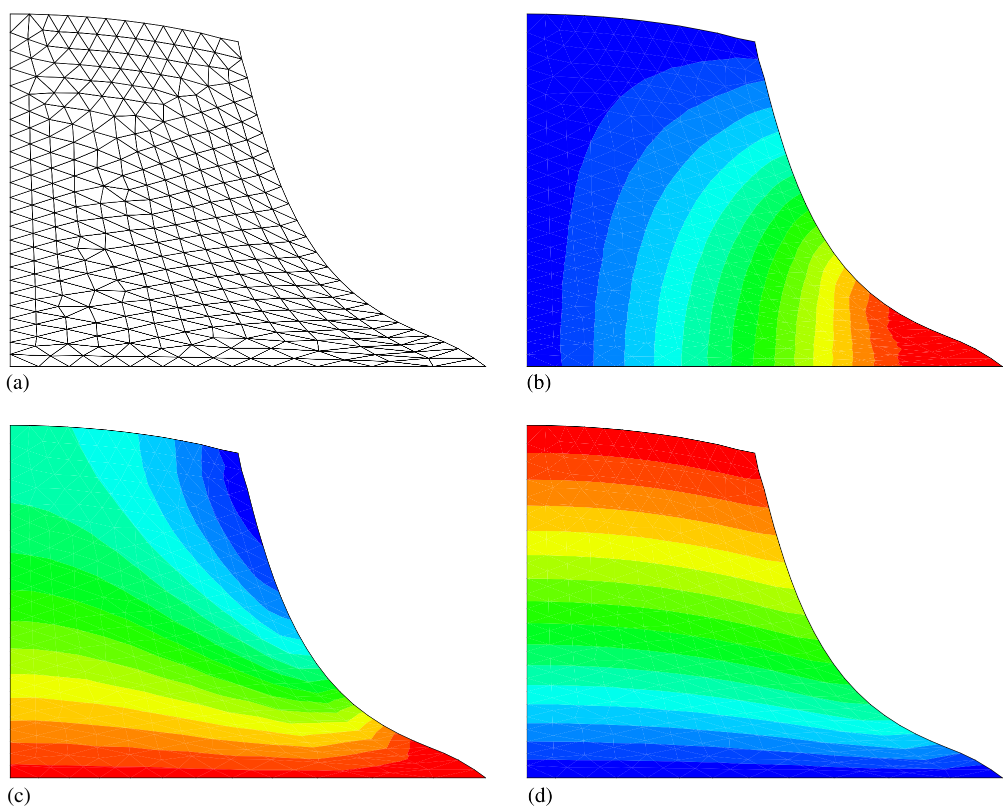

Figure 2. Broken dam problem. Mesh and contours after $t = 2.0$: (a) mesh; (b) u_1 velocity contours; (c) u_2 velocity contours; and (d) pressure contours.

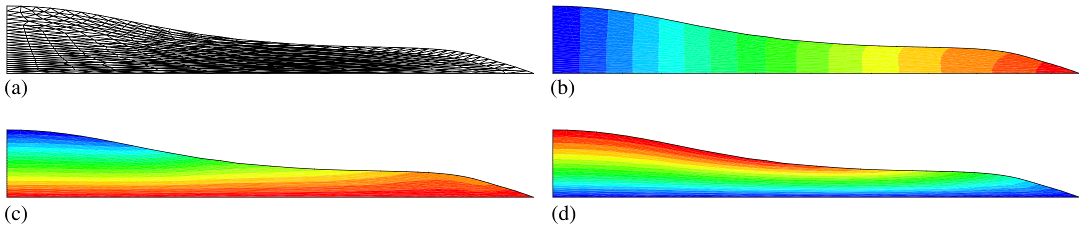

Figure 3. Broken dam problem. Mesh and contours after $t = 5.0$: (a) mesh; (b) u_1 velocity contours; (c) u_2 velocity contours; and (d) pressure contours.

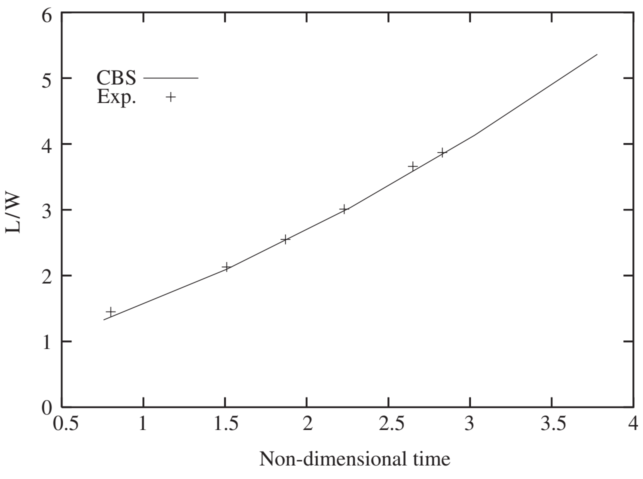

Figure 4. Broken dam problem. Comparison of numerical results with experimental data [5].

reported experimental data. The non-dimensional time in the horizontal coordinate is calculated as $t\sqrt{2g/\text{W}}$.

In Figure 5 we show the area error with respect to time. The area error is calculated as the difference between the initial total area and the total area value with respect to time. The initial area is 24.5. Only a very small change in the total area with respect to time is noted. This error is especially pronounced close to a time of 3. The maximum percentage of error is less than 0.3%.

5.2. Solitary wave propagation

We now consider an example of a solitary wave propagation between two solid vertical walls. In this problem we employ the described ALE procedure. Figure 6 shows the problem definition. It consists of a liquid with free surface constrained within three walls, two vertical walls and one bottom horizontal wall. The total horizontal length of the domain is $16d$ and $d = 1$. The gravity direction is downward vertical with $g = 9.81$. The viscosity of the fluid was assumed to be 0.01. The time step employed was 0.025 which is very close to the stability limit of the scheme. A time step value close to the stability limit is advisable to obtain a stable pressure solution.

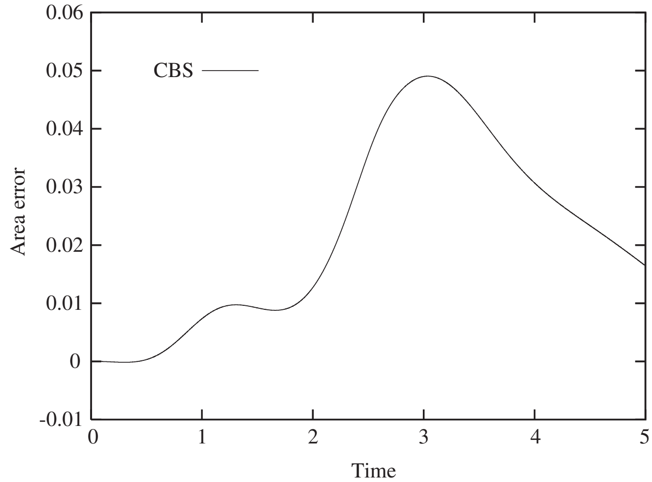

Figure 5. Area error distribution with respect to time.

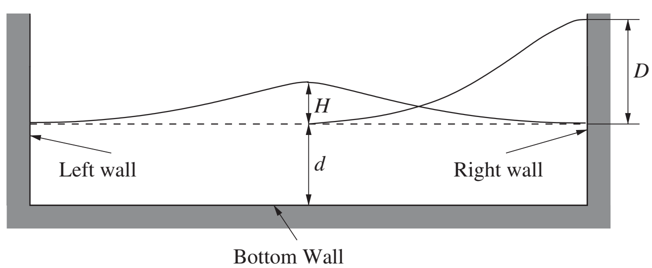

Figure 6. Solitary wave propagation. Problem definition.

The walls of the problem are assumed to be slip walls and initial conditions are calculated based on the work presented by Laitone [39] for an infinite domain. The relationships for total wave height, velocity components and pressure are given as

$$h = d + H\,\mathrm{sech}^2\left[\sqrt{\frac{3H}{4d^3}}\,(x_1 - ct)\right] \tag{9}$$

$$u_1 = \sqrt{gd}\,\frac{H}{d}\,\mathrm{sech}^2\left[\sqrt{\frac{3H}{4d^3}}(x_1 - ct)\right] \tag{10}$$

$$u_2 = \sqrt{3gd}\left(\frac{H}{d}\right)^{3/2}\left(\frac{x_2}{d}\right)\mathrm{sech}^2\left[\sqrt{\frac{3H}{4d^3}}(x_1 - ct)\right]\tanh\left[\sqrt{\frac{3H}{4d^3}}(x_1 - ct)\right] \tag{11}$$

and

$$p = \rho g(h - x_2) \tag{12}$$

In the above equation c is given as

$$\frac{c}{\sqrt{gd}} = 1 + \frac{1}{2}\frac{H}{d} - \frac{3}{20}\left(\frac{H}{d}\right)^2 + O\left(\frac{H}{d}\right)^3 \tag{13}$$

The initial solution and geometry are generated by substituting $t = 0$ into Equations (9)–(11). The mesh smoothing procedure is carried out by recalculating the coordinates of the nodes as discussed in Sections 3 and 4.

In Figures 7 and 8 we show the meshes and the velocity vectors at various time levels for $H/d = 0.3$. The total number of elements and nodes are unchanged during the calculation, they are 3838 and 2092. The semi-implicit form of the CBS scheme was again used in the calculations. As seen the wave reaches a maximum height at the right wall around a time,

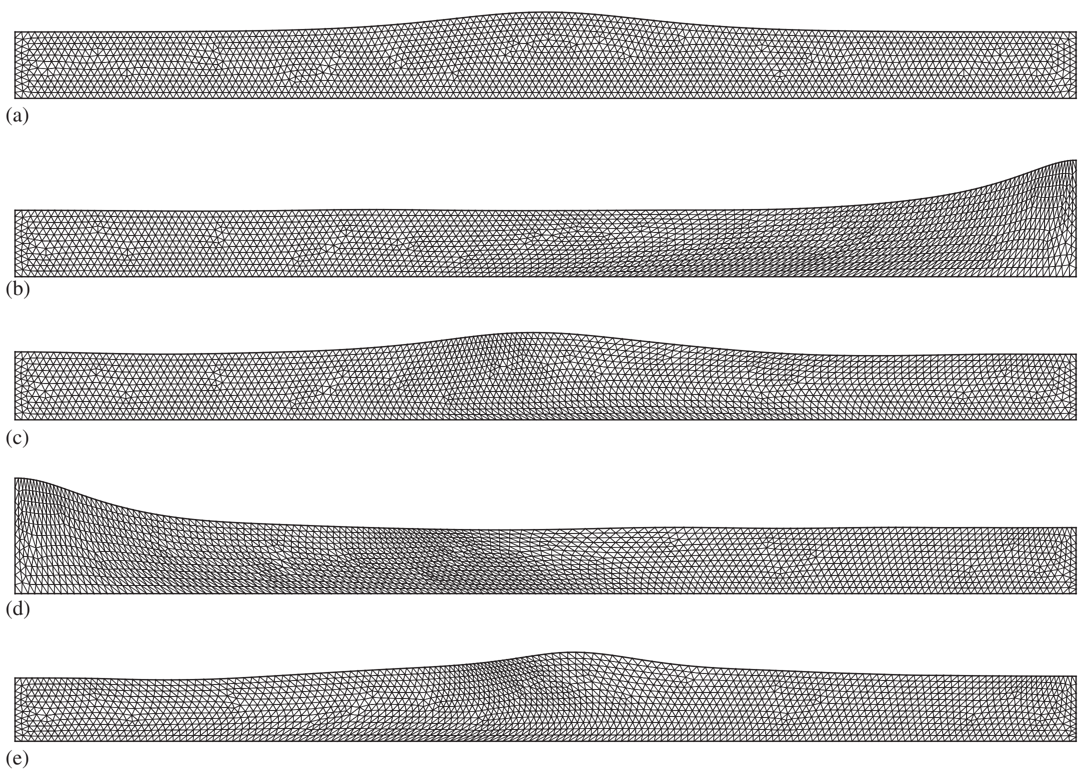

Figure 7. Solitary wave propagation. Meshes at various time levels: (a) $t = 0.0$; (b) $t = 2.3$; (c) $t = 4.6$; (d) $t = 6.9$; and (e) $t = 9.2$.

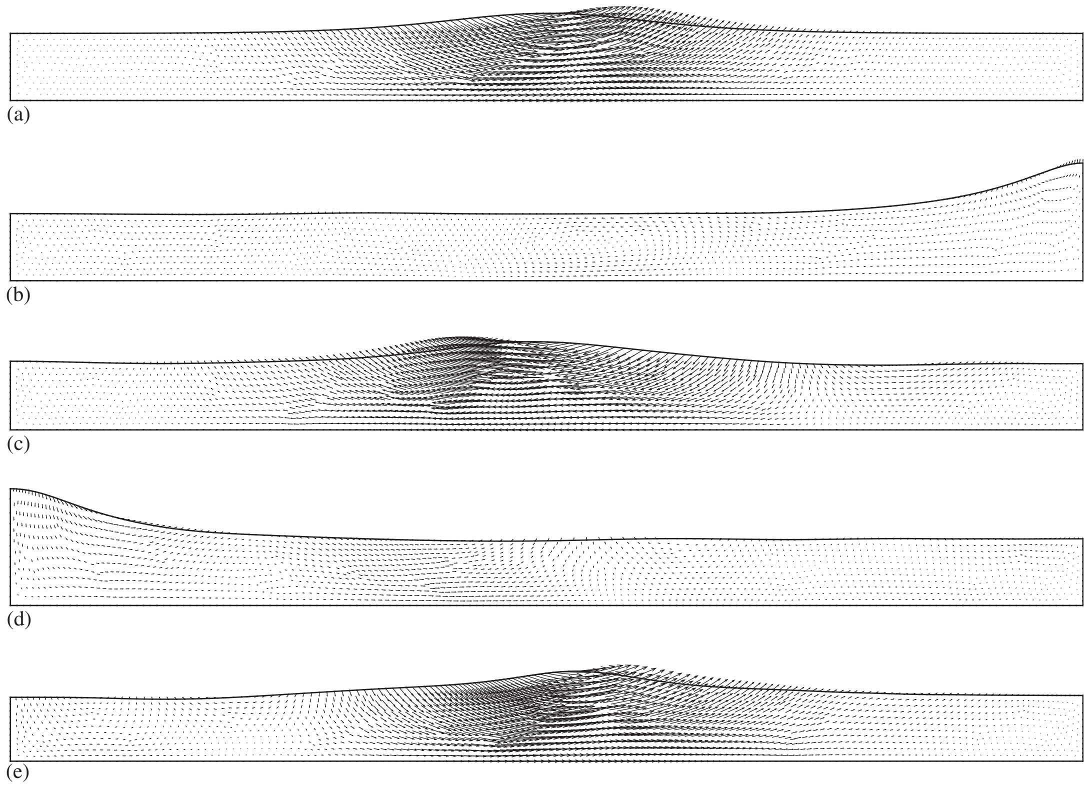

Figure 8. Solitary wave propagation. Velocity vector distribution at various time levels: (a) $t = 0.0$; (b) $t = 2.3$; (c) $t = 4.6$; (d) $t = 6.9$; and (e) $t = 9.2$.

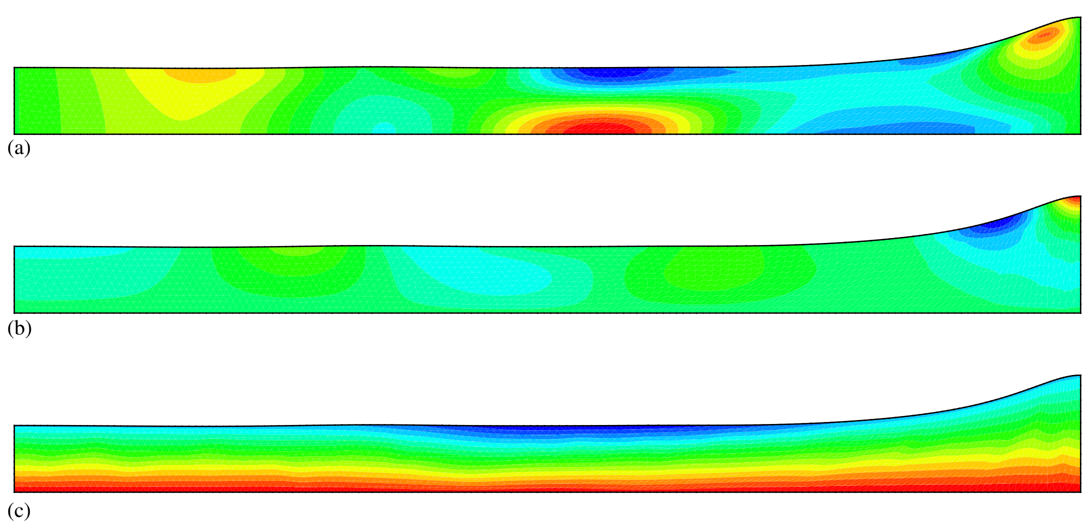

Figure 9. Solitary wave propagation. u_1, u_2 and p distribution at $t = 2.3$: (a) u_1; (b) u_2; and (c) p.

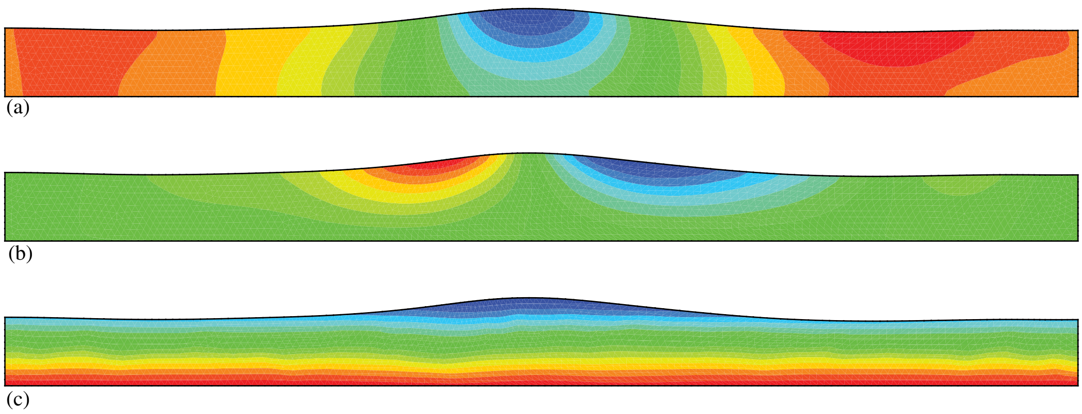

Figure 10. Solitary wave propagation. u_1, u_2 and p distribution at $t = 4.6$: (a) u_1; (b) u_2; and (c) p.

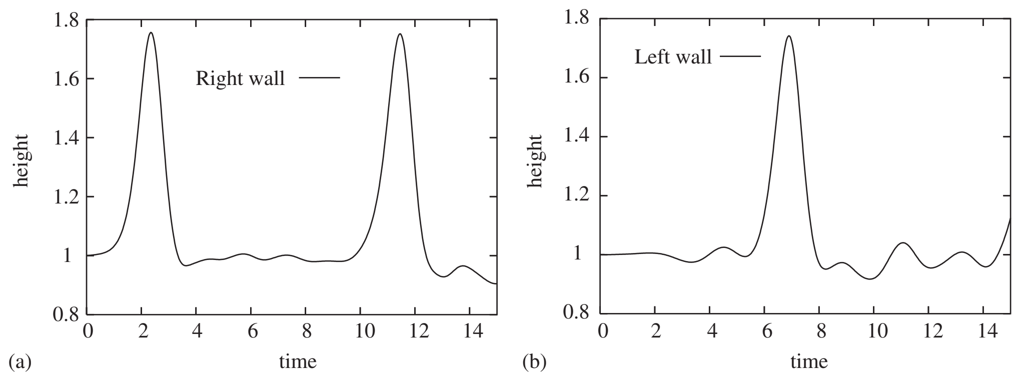

Figure 11. Wave heights with respect to time on the right and left side walls: (a) right wall; and (b) left wall.

$t = 2.3$ and returning to the initial position at around $t = 4.6$. This values are in very close agreement with other reported numerical results [23].

Figures 9 and 10 show the contours of u_1 and u_2 velocity components and pressure at $t = 2.3$ and 4.6. As seen the contours show no sign of spatial oscillations and smooth everywhere. These figures are in qualitative agreement with the available numerical solution [17].

Figure 11 shows the fluid height variation at right and left walls with time. As seen the first peak at the right wall is reached around a time $t = 2.3$ and at left wall it reached at 6.9.

Figure 12 shows the comparison of maximum height reached D against the experimental data of Maxworthy [40]. As seen the agreement between the numerical and experimental data is quite good upto $H = 0.4$. Beyond 0.4 the accuracy deteriorates. This is one of the limitations of employing mesh smoothing and keeping the same number of elements through out the calculation. An appropriate mesh regeneration algorithm may increase the accuracy further, however, at the expense of additional computational cost.

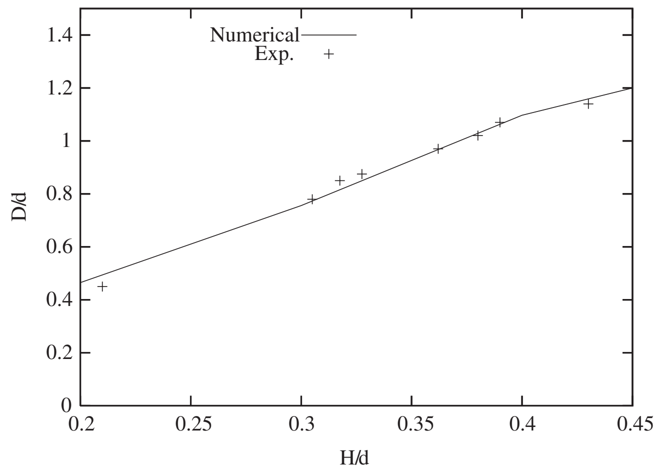

Figure 12. Solitary wave propagation. Comparison of wave heights with experimental data [40].

6. CONCLUDING REMARKS

In this note we have described the use of the CBS scheme for the solution of incompressible free surface flows in the arbitrary Lagrangian Eulerian (ALE) framework. The ALE method described was based on a simple mesh smoothing approach, which may not be a perfect option for larger wave amplitudes. For solving breaking wave problems recently published meshless finite element method of Idelsohn *et al*. [41–43] and the Smooth Particle Hydrodynamics (SPH) method [44] should be consulted.

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